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Approximate Selection Theorems in H-Spaces with Applications*

Xian Wu and Feng Li

Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650092

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In the present paper, some approximate selection theorems, an almost fixed point theorem, a fixed point theorem, and an existence theorem for solution of generalized quasi-variational inequality in H-spaces are obtained. © 1999 Academic Press

Key Words: l.c.-space; locally convex H-space; quasi-lower semicontinuous multi-valued mapping; acyclic set.

1. INTRODUCTION AND PRELIMINARIES

In 1956, Michael [8] first researched the approximate selection problem and obtained the following result:

THEOREM 1.1. *Let Y be a normed linear space, and let X be a paracompact topological space. If $F: X \to 2^Y$ is a lower semicontinuous set-valued mapping with nonempty convex values, then for each convex neighborhood V of the origin of Y, there exists a continuous mapping $f: X \to Y$ such that*

$$f(x) \in F(x) + V, \qquad \forall x \in X.$$

By Theorem 1.1 Michael [8] proved a celebrated continuous selection theorem (i.e., Michael selection theorem). Since then, the Michael selection theorem and the approximate selection theorem have become useful tools in nonlinear analysis.

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In 1981, Hadžić [5] researched the approximate selection problem and the almost fixed point problem for a uniformly u-continuous multivalued mapping in topological vector spaces. She obtained the following approximate selection theorem:

THEOREM 1.2. *Let X be a topological vector space, let $\mathscr{U}$ be the family of all neighborhoods of zero in X, and let K be a paracompact convex subset of X. Let $F\colon K \to 2^K$ be a uniformly u-continuous multivalued mapping with nonempty closed convex values. If for each $V \in \mathscr{U}$, there exists $U \in \mathscr{U}$ such that*

$$\mathrm{co}(U \cap (F(K) - F(K))) \subset V,$$

then for each $V \in \mathscr{U}$, there exists a continuous mapping $g\colon K \to K$ such that

$$g(x) \in F(x) + V, \qquad \forall x \in K.$$

In 1991, Horvath [3] generalized Theorem 1.1 to H-spaces. He established the following approximate selection theorem:

THEOREM 1.3. *Let $(Y, \{\Gamma_A\})$ be an l.c.-space with a uniformity $\mathscr{U}$ and X a paracompact topological space. If $F\colon X \to 2^Y$ is a lower semicontinuous set-valued mapping with nonempty H-convex values, then for each $V \in \mathscr{U}$, there exists a continuous mapping $f\colon X \to Y$ such that*

$$F(x) \cap V(f(x)) \neq \varnothing, \qquad \forall x \in X.$$

In the papers [17] and [6], Zheng and Hadžić researched the approximate selection problem for sublower semicontinuous multivalued mappings in topological vector spaces and almost fixed point problems for a lower semicontinuous multivalued mapping with the generalized Zima type condition in H-spaces with uniformity, respectively.

In the present paper, our purposes are to establish some new approximate selection theorems, a new almost fixed point theorem for quasi-lower semicontinuous multivalued mapping, and a new existence theorem for solutions of generalized quasi-variational inequality in H-spaces.

In order to establish our main results, we give some concepts and notations, (see also, [2–4]).

Let X be a topological space and let $\mathscr{F}(X)$ be the family of all nonempty finite subsets of X. Let $\{\Gamma_A\}$ be a family of some nonempty contractible subsets of X indexed by $A \in \mathscr{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an H-space. Given an H-space $(X, \{\Gamma_A\})$, a nonempty subset D of X is called

(1) H-convex if $\Gamma_A \subset D$ for all $A \in \mathscr{F}(D)$;

(2) weakly H-convex if $\Gamma_A \cap D$ is nonempty contractible for each $A \in \mathscr{F}(D)$;

(3) H-compact in X if for each $A \in \mathscr{F}(X)$, there exists a compact weakly H-convex subset D_A of X such that $D \cup A \subset D_A$.

For a nonempty subset K of X, we define the H-convex hull of K, denoted by H-co K, as

$$H\text{-co } K = \bigcap \{D \subset X\colon D \text{ is } H\text{-convex and } K \subset D\}.$$

If $K = \emptyset$, we always consider H-co $K = \emptyset$, (see also, [13]).

An H-space $(X, \{\Gamma_A\})$ is called

(1) a locally convex H-space if X is a uniform space and if there exists a base $\{V_i\colon i \in I\}$ for the uniform structure $\mathscr{U}$ such that for each $i \in I$, $V_i(x) = \{y \in X\colon (y, x) \in V_i\}$ is H-convex for each $x \in X$; ([16])

(2) an l.c.-space (see [3]) if X is a uniform space and if there exists a base $\{V_i\colon i \in I\}$ for the uniform structure such that for each $i \in I$, the set $\{x \in X\colon \; E \cap V_i[x] \neq \emptyset\}$ is H-convex whenever E is H-convex, where $V_i[x] = \{y \in X\colon (x, y) \in V_i\}$.

Remark. The concept of an l.c.-space is different from a locally convex H-space. But an l.c.-space $(X, \{\Gamma_A\})$ with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$ must be a locally convex H-space. Otherwise, a nonempty convex subset X of a locally convex topological vector space must be an l.c.-space with $\Gamma_A = \text{co } A$ for all $A \in \mathscr{F}(X)$, and hence $(X, \{\text{co } A\})$ must be a locally convex H-space.

Let X be a topological space. We denote by 2^X the family of all subsets of X. If $A \subset X$ we shall denote by cl(A) and int(A) the closure and interior of A, respectively.

A topological space is said to be acyclic if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic.

Let X, Y be two topological spaces and let $T\colon X \to 2^Y$ be a multivalued mapping. T is said to be upper semicontinuous (resp., lower semicontinuous) if for each $x \in X$ and for each open set $V \subset Y$ with $T(x) \subset V$ (resp., $T(x) \cap V \neq \emptyset$), there exists an open neighborhood U of x such that $T(z) \subset V$ (resp., $T(z) \cap V \neq \emptyset$) for each $z \in U$. For each $y \in Y$, we denote $T^{-1}(y) = \{x \in X\colon \; y \in T(x)\}$, which is called a lower section of T. If $(Y, \{\Gamma_A\})$ is an H-space with a uniform structure $\mathscr{U}$, then T is said to be quasi-lower semicontinuous if for each $x \in X$ and for each $V \in \mathscr{U}$, there exists a point $y \in T(x)$ and a neighborhood $U(x)$ of x such that for each $z \in U(x)$, $T(z) \cap V(y) \neq \emptyset$. When Y is a topological vector space, the following concept is given in [17]:

T is said to be sublower semicontinuous if for each $x \in X$ and for each neighborhood V of 0 in Y, there exist a point $y \in T(x)$ and a neighborhood $U(x)$ of x in X such that for each $z \in U(x)$, $y \in T(z) + V$.

The following propositions show that the quasi-lower semicontinuity is weaker than lower semicontinuity and sublower semicontinuity.

PROPOSITION 1.1. *Let X be a topological space and let $(Y, \{\Gamma_A\})$ be an H-space with a uniform structure $\mathscr{U}$. If $T\colon X \to 2^Y$ is a lower semicontinuous set-valued mapping with nonempty values, then T is quasi-lower semicontinuous.*

Proof. For each $x \in X$ and for each $V \in \mathscr{U}$, there exists symmetric element $W \in \mathscr{U}$ such that $W \subset V$. Because $T(x) \neq \varnothing$, there exists a point $y \in T(x)$, and hence $T(x) \cap W(y) \neq \varnothing$. By the lower semicontinuity of T there is a neighborhood $U(x)$ of x such that for each $z \in U(x)$, $T(z) \cap W(y) \neq \varnothing$ so that $T(z) \cap V(y) \neq \varnothing$. This shows that T is quasi-lower semicontinuous.

Throughout the paper, all spaces are assumed to be Hausdorff. For each topological vector space Y, let $\mathscr{U}$ be the family of all neighborhoods of 0 in Y. For each $U \in \mathscr{U}$, let $\tilde{U} = \{(x, y) \in X \times X\colon\ y - x \in U\}$. It is clear that $(X, \{\text{coA}\})$ $(A \in \mathscr{F}(X))$ is an H-space with uniformity $\tilde{\mathscr{U}} = \{\tilde{U}\colon U \in \mathscr{U}\}$.

Remark. The example in Section 3 shows that there are quasi-lower semicontinuous mulivalued mappings, which need not be lower semicontinuous.

PROPOSITION 1.2. *Let X be a topological space, let Y be a topological vector space, and let $T\colon X \to 2^Y$ be a set-valued mapping. Then T is sublower semicontinuous iff T is quasi-lower semicontinuous.*

Proof. If T is sublower semicontinuous, then for each $x \in X$ and for each $\tilde{V} \in \tilde{\mathscr{U}}$, there exists a point $y \in T(x)$ and a neighborhood $U(x)$ of x in X such that $y \in T(z) + V$ for each $z \in U(x)$. Consequently, for each $z \in U(x)$, there is a point $y^* \in T(z)$ such that $(y - y^*) \in V$, and hence $y^* \in T(z) \cap \tilde{V}(y)$. This shows that T is quasi-lower semicontinuous. On the other hand, if T is quasi-lower semicontinuous, then for each $x \in X$, and for each neighborhood V of 0 in Y, there exist a point $y \in T(x)$ and a neighborhood $U(x)$ of x in X such that $T(z) \cap \tilde{V}(y) \neq \varnothing$ for each $z \in U(x)$. Consequently, for each $z \in U(x)$, there is a point $y^* \in T(z)$ such that $(y - y^*) \in V$, and hence $y \in T(z) + V$. It shows that T is sublower semicontinuous.

2. TWO LEMMAS

We begin with the following lemmas.

LEMMA 1. *Let X be a topological space, let $(Y, \mathscr{U})$ be a uniform space, and let $\mathscr{B}$ be a basis of $\mathscr{U}$. Let $T\colon X \to 2^Y$ be a multivalued mapping. If $\bar{x} \in X$ and $\bar{y} \in Y$ such that*

$$\bar{y} \in \bigcap_{V \in \mathscr{B}} V\big(\overline{T}(\bar{x})\big),$$

(*where*, $\overline{T}(\bar{x}) = \{y \in Y\colon\ (\bar{x}, y) \in \mathrm{cl}(\mathrm{graph}\, T)\}$, $V(\overline{T}(\bar{x})) = \bigcup_{y \in \overline{T}(\bar{x})} V(y)$), *then $\bar{y} \in \overline{T}(\bar{x})$.*

Proof. If $\bar{y} \notin \overline{T}(\bar{x})$, then $(\bar{x}, \bar{y}) \notin \mathrm{cl}(\mathrm{graph}\, T)$. Consequently, there exist an open neighborhood U of $\bar{x}$ and an open symmetric element $V \in \mathscr{U}$ such that

$$(U \times V(\bar{y})) \cap \mathrm{cl}(\mathrm{graph}\, T) = \varnothing.$$

Take an element $W \in \mathscr{B}$ such that $W \subset V$. Because $\bar{y} \in W(\overline{T}(\bar{x})) \subset V(\overline{T}(\bar{x}))$, there is a point $\bar{z} \in \overline{T}(\bar{x})$ such that $\bar{y} \in V(\bar{z})$, and hence $\bar{z} \in V(\bar{y})$ because V is symmetric. Hence,

$$(\bar{x}, \bar{z}) \in (U \times V(\bar{y})) \cap \mathrm{cl}(\mathrm{graph}\, T).$$

It contradicts that $(U \times V(\bar{y})) \cap \mathrm{cl}(\mathrm{graph}\, T) = \varnothing$. Therefore, $\bar{y} \in \overline{T}(\bar{x})$.

LEMMA 2 ([16]). *Let $(X, \{\Gamma_A\})$ be a Hausdorff locally convex H-space and let D be an H-compact subset of X. If $T\colon X \to 2^D$ is an upper semicontinuous multivalued mapping with closed acyclic values, then there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.*

Now, we establish our main results.

3. APPROXIMATE SELECTION THEOREMS

THEOREM 3.1. *Let $(Y, \{\Gamma_A\})$ be an l.c.-space with a uniformity $\mathscr{U}$ and let X be a paracompact topological space. If $F\colon X \to 2^Y$ is a quasi-lower semicontinuous set-valued mapping with H-convex values, then for each $V \in \mathscr{U}$, there exists a continuous mapping $f\colon X \to Y$ such that*

$$F(x) \cap V(f(x)) \neq \varnothing, \qquad \forall x \in X.$$

Furthermore, if X is a precompact uniform space or compact topological space, there is a finite subset $A_0 \subset X$ such that $f(X) \subset \Gamma_{A_0}$.

Proof. Because $(Y, \{\Gamma_A\})$ is an l.c.-space with a uniform structure $\mathscr{U}$, there exists a base $\mathscr{V}$ of $\mathscr{U}$ such that for each $V \in \mathscr{V}$, the set $\{y \in Y: E \cap V[y] \neq \emptyset\}$ is H-convex whenever E is H-convex in Y. For each $V \in \mathscr{U}$, there exists an element $V_1 \in \mathscr{V}$ and two open symmetric elements $W \in \mathscr{U}$, $W_1 \in \mathscr{U}$ such that $W \subset V_1 \subset W_1 \subset V$. For each $x \in X$, because $F: X \to 2^Y$ is a quasi-lower semicontinuous set-valued mapping, we take a fixed point $y(x) \in F(x)$ and a fixed open neighborhood $U(x)$ of x such that

$$F(z) \cap W(y(x)) \neq \emptyset, \qquad \forall z \in U(x).$$

Because X is paracompact, there exists a locally finite open refinement $\mathscr{R}$ of the family $\{U(x): x \in X\}$. For each $O \in \mathscr{R}$, take a fixed $z(O) \in X$ such that $O \subset U(z(O))$. Consequently, $y(z(O)) \in F(z(O))$ and

$$F(u) \cap W(y(z(O))) \neq \emptyset, \qquad \forall u \in U(z(O)).$$

Let $\eta(O) = y(z(O))$. Then $\eta: \mathscr{R} \to Y$ is a mapping. By Theorem 1 in [3] there exists a continuous mapping $f: X \to Y$ such that

$$f(x) \in \Gamma_{\{\eta(O):\, O \in \sigma(x, \mathscr{R})\}},$$

for all $x \in X$, where $\sigma(x, \mathscr{R}) = \{O \in \mathscr{R}: x \in O\}$.

For each $x \in X$ and for each $O \in \sigma(x, \mathscr{R})$, because $x \in O \subset U(z(O))$,

$$F(x) \cap W(y(z(O))) \neq \emptyset.$$

Consequently, $F(x) \cap V_1[\eta(O)] \neq \emptyset$. Hence $\{\eta(O): O \in \sigma(x, \mathscr{R})\} \subset \{y \in Y: F(x) \cap V_1[y] \neq \emptyset\}$. Because again $F(x)$ is H-convex and $V_1 \in \mathscr{V}$,

$$f(x) \in \Gamma_{\{\eta(O):\, O \in \sigma(x, \mathscr{R})\}} \subset \{y \in Y: F(x) \cap V_1[y] \neq \emptyset\},$$

and hence,

$$\begin{aligned} F(x) \cap V(f(x)) &\supset F(x) \cap W_1(f(x)) = F(x) \cap W_1[f(x)] \\ &\supset F(x) \cap V_1[f(x)] \neq \emptyset. \end{aligned}$$

This completes the proof of first part of Theorem 3.1.

If X is a precompact uniform space or a compact topological space, $\mathscr{R}$ can be chosen finite. Take $A_0 = \{\eta(O): O \in \mathscr{R}\}$, then A_0 is a finite subset of X and $f(X) \subset \Gamma_{\{\eta(O):\, O \in \mathscr{R}\}} = \Gamma_{A_0}$.

Remark. Theorem 3.1 contains Theorem 1.3 (i.e., Proposition 1 of Horvath [3]) as a special case. In addition, the following example shows that Theorem 3.1 is a true generalization of Theorem 1.3.

EXAMPLE. Let R be the set of all real numbers and $R^+ = \{x \in R: x > 0\}$. Then R^+ is paracompact. For each $x \in R^+$, let

$$T(x) = \begin{cases} [x/2, 2x], & \text{if } 0 < x \leq 1, \\ [x, 2x], & \text{if } x > 1. \end{cases}$$

Then $T: R^+ \to 2^R$ is sublower semicontinuous. Indeed, for each $x \in R^+$ and for each neighborhood V of 0 in R,

(1) if $x = 1$, then $T(1) = [\frac{1}{2}, 2]$. Take $y_0 = \frac{3}{2}$ and $U(1) = (\frac{3}{4}, \frac{5}{4})$. Then $y_0 \in T(1), U(1)$ is a neighborhood of 1, and $y_0 \in T(z) \subset T(z) + V$ for all $z \in U(1)$;

(2) if $0 < x < 1$, then $T(x) = [x/2, 2x]$. Take $y_0 = x \in T(x)$ and $U(x) = (\frac{3}{4}x, \varepsilon x)$, where $1 < \varepsilon < \min\{2, 1/x\}$. Then $U(x)$ is a neighborhood of x and $y_0 \in T(z) \subset T(z) + V$ for all $z \in U(x)$;

(3) if $x > 1$, then $T(x) = [x, 2x]$. Take $y_0 = \frac{3}{2}x \in T(x)$ and $U(x) = (\delta x, \frac{3}{2}x)$, where $\max\{\frac{3}{4}, 1/x\} < \delta < 1$. Then $U(x)$ is a neighborhood of x and $y_0 \in T(z) \subset T(z) + V$ for all $z \in U(x)$.

But $T: R^+ \to 2^R$ is not lower semicontinuous. In fact, for $x = 1$, we take $V = (\frac{1}{2}, \frac{5}{8})$. Then V is an open set in R and $T(1) \cap V \neq \emptyset$. For any neighborhood U of 1 in R^+, there is a point $x_0 \in U$ such that $x_0 > 1$, and hence $T(x_0) \cap V = \emptyset$. This shows that $T: R^+ \to 2^R$ is not lower semicontinuous. Hence Theorem 3.1 is suitable, but Theorem 1.3 is not suitable, to assure the existence of an approximate selection of T.

Remark. Applying Theorem 3.1 we know that in Theorem 3 of Horvath [3], the condition "T is lower semicontinuous" may be weakened as "T is quasi-lower semicontinuous."

COROLLARY 3.2. *Let Y be a locally convex topological vector space and let X be a paracompact topological space. If $F: X \to 2^Y$ is a sublower semicontinuous set-valued mapping with nonempty convex values, then for each neighborhood V of 0 in Y, there exists a continuous mapping $f: X \to Y$ such that*

$$f(x) \in F(x) + V, \qquad \forall x \in X.$$

Furthermore, if X is a precompact uniform space or a compact topological space, there is a finite subset $A_0 \subset X$ such that $f(X) \subset \operatorname{co} A_0$.

Proof. For each neighborhood V of 0 in Y, by Proposition 1.2 and Theorem 3.1 there exists a continuous mapping $f: X \to Y$ such that

$$F(x) \cap \tilde{V}(f(x)) \neq \emptyset, \qquad \forall x \in X.$$

Furthermore, if X is a precompact uniform space or a compact topological space, there is a finite subset $A_0 \subset X$ such that $f(X) \subset \Gamma_{A_0}$.

But $F(x) \cap \tilde{V}(f(x)) \neq \emptyset$ implies that $f(x) \in F(x) + V$. Hence Corollary 3.2 is proved.

THEOREM 3.3. *Let $(Y, \{\Gamma_A\})$ be an H-space with a uniformity $\mathscr{U}$, let X be a paracompact topological space, and let $F\colon X \to 2^Y$ be a set-valued mapping. If for each $V \in \mathscr{U}$, there exists a continuous mapping $f\colon X \to Y$ such that*

$$F(x) \cap V(f(x)) \neq \emptyset, \qquad \forall x \in X,$$

then $F\colon X \to 2^Y$ is quasi-lower semicontinuous.

Proof. For each $V \in \mathscr{U}$, there exist open symmetric elements $W_1, W \in \mathscr{U}$ such that $W_1 \subset V$ and $W \circ W \circ W \subset W_1$. For the W, there exists a continuous mapping $f\colon X \to Y$ such that

$$F(x) \cap W(f(x)) \neq \emptyset, \qquad \forall x \in X.$$

Consequently, for each $x \in X$, there is a point $y \in F(x)$ such that $y \in W(f(x))$. By the continuity of f there is a neighborhood $U(x)$ of x such that $f(z) \in W(f(x))$ for all $z \in U(x)$.

For each $z \in U(x)$, because $F(x) \cap W(f(z)) \neq \emptyset$, there exists a point $y^* \in F(z)$ such that $y^* \in W(f(z))$. Consequently, $(y^*, y) \in W \circ W \circ W \subset W_1 \subset V$. Hence $y^* \in F(z) \cap V(y)$. This shows that $F\colon X \to 2^Y$ is quasi-lower semicontinuous.

COROLLARY 3.4. *Let Y be a topological vector space, let X be a paracompact topological space and let $F\colon X \to 2^Y$ be a set-valued mapping. If for each neighborhood V of 0 in Y, there exists a continuous mapping $f\colon X \to Y$ such that*

$$f(x) \in F(x) + V, \qquad \forall x \in X,$$

then $F\colon X \to 2^Y$ is sublower semicontinuous.

Proof. For each $\tilde{V} \in \tilde{\mathscr{U}}$, because V is a neighborhood of 0 in Y, there exists a continuous mapping $f\colon X \to Y$ such that

$$f(x) \in F(x) + V, \qquad \forall x \in X,$$

i.e.,

$$F(x) \cap V(f(x)) \neq \emptyset, \qquad \forall x \in X.$$

By Theorem 3.3 and Proposition 1.2 we know that $F\colon X \to 2^Y$ is sublower semicontinuous.

Remark. Summing-up Corollaries 3.2 and 3.4 we have Corollary 3.5 (i.e., Theorem 2.1 in [17]):

COROLLARY 3.5 [17]. *Let Y be a locally convex topological vector space, let X be a paracompact topological space, and let $F: X \to 2^Y$ be a set-valued mapping with nonempty convex values. the F is sublower semicontinuous iff for each neighborhood V of 0 in Y, there exists a continuous mapping $f: X \to Y$ such that*

$$f(x) \in F(x) + V, \qquad \forall x \in X.$$

In order to state other results we give the following concepts.

DEFINITION 3.1. Let X be a topological space and let Y be a uniform space with a uniformity $\mathscr{U}$. We say that set-valued mappings $F, T: X \to 2^Y$ are topologically separated iff for each $x \in X$ there exists a neighborhood $U(x)$ of x and an element $V \in \mathscr{U}$ such that $F(U(x)) \cap V(T(x)) = \emptyset$.

Remark. Definition 3.1 generalizes a corresponding concept in [17].

DEFINITION 3.2. An H-space $(Y, \{\Gamma_A\})$ is called an a.l.c. metric space iff Y is metrizable by a metric d and for each $\varepsilon > 0$, the set $\{y \in Y: d(y, E) < \varepsilon\}$ is an H-convex set whenever E is an H-convex subset of Y.

Remark. An l.c. metric space (see, [3]) must be an a.l.c. metric space.

THEOREM 3.6. *Let X be a compact topological space and let $(Y, \{\Gamma_A\})$ be an l.c.-space with a uniformity $\mathscr{U}$. If $F, T: X \to 2^Y$ are two set-valued mappings such that*

(i) *F and T are topologically separated,*

(ii) *T is upper semicontinuous,*

(iii) *F is quasi-lower semicontinuous and $F(x)$ is nonempty H-convex for each $x \in X$, then for each $V \in \mathscr{U}$, there exists a continuous mapping $f: X \to Y$ such that*

$$F(x) \cap V(f(x)) \neq \emptyset \quad \textit{and} \quad f(x) \notin T(x), \qquad \forall x \in X.$$

Furthermore, there is a finite subset $A_0 \subset X$ such that $f(X) \subset \Gamma_{A_0}$.

Proof. For each fixed $V \in \mathscr{U}$, by (i), for each $x \in X$ there exist an open neighborhood $U(x)$ of x and an open symmetric element $V_x \in \mathscr{U}$ such that $V_x \subset V$ and $F(U(x)) \cap V_x(T(x)) = \emptyset$.

Take an open symmetric element $W_x \in \mathscr{U}$ such that $W_x \circ W_x \subset V_x$. By (ii) there is an open neighborhood $O(x)$ of x in X such that $T(O(x)) \subset W_x(T(x))$ and $O(x) \subset U(x)$. Because X is compact, there exists a finite subset $\{x_1, x_2, \dots, x_n\} \subset X$ and an open symmetric element $W \in \mathscr{U}$ such that a $X = \bigcup_{i=1}^{i=n} O(x_i)$ and $W \subset \bigcap_{i=1}^{i=n} W_{x_i}$. By (iii) and Theorem 3.1 there is a continuous mapping f: $X \to Y$ such that

$$F(x) \cap W(f(x)) \neq \varnothing, \qquad \forall x \in X,$$

and there is a finite subset $A_0 \subset X$ such that $f(X) \subset \Gamma_{A_0}$.

For each $x \in X$, we assert $W(F(x)) \cap T(x) = \varnothing$ to be true. Otherwise, there exist a point $y \in T(x)$ and a point $z \in F(x)$ such that $(z, y) \in W$. We may assume $x \in O(x_k)$ ($k \in \{1, 2, \dots, n\}$). Hence $T(x) \subset W_{x_k}(T(x_k))$ so that $y \in W_{x_k}(T(x_k))$. Consequently, there is $y_k \in T(x_k)$ such that $(y, y_k) \in W_{x_k}$. Therefore, $(z, y_k) \in W_{x_k} \circ W_{x_k} \subset V_{x_k}$ and thus $z \in V_{x_k}(y_k) \subset V_{x_k}(T(x_k))$. On the other hand, $z \in F(x) \subset F(O(x_k)) \subset F(U(x_k))$. Hence $z \in F(U(x_k)) \cap V_{x_k}(T(x_k)) = \varnothing$. It is a contradiction. This shows that $W(F(x)) \cap T(x) = \varnothing$. Hence $f(x) \notin T(x)$ for all $x \in X$. This completes the proof.

Remark. Theorem 3.6 improves and extends Theorem 2.2 of [17] to H-spaces.

THEOREM 3.7. *Let X be a paracompact topological space and let $(Y, \{\Gamma_A\})$ be an a.l.c. metric space with the metric d. If F, T: $X \to 2^Y$ are two set-valued mappings such that*

(i) *F and T are topologically separated,*

(ii) *T is upper semicontinuous,*

(iii) *F is quasi-lower semicontinuous and $F(x)$ is H-convex for each $x \in X$, then for each $\varepsilon > 0$, there exists a continuous mapping f: $X \to Y$ such that*

$$F(x) \cap V_\varepsilon(f(x)) \neq \varnothing \quad \text{and} \quad f(x) \notin T(x), \qquad \forall x \in X,$$

where $V_\varepsilon = \{(y, z) \in Y \times Y: d(y, z) < \varepsilon\}$.

Proof. For each fixed $\varepsilon > 0$, by (i) and (ii), for each $x \in X$ there exists a neighborhood $U(x)$ of x and an $\eta(x) > 0$ such that $\eta(x) < \varepsilon$, $F(U(x)) \cap V_{\eta(x)}(T(x)) = \varnothing$ and $T(U(x)) \subset V_{\eta(x)/2}(Tx)$. Let $\varepsilon(x) = \eta(x)/2$. For each $y \in U(x)$, we assert $F(y) \cap V_{\varepsilon(x)}(Ty) = \varnothing$. Otherwise, there exist a point $p \in T(y)$ and a point $z \in F(y)$ such that $d(p, z) < \varepsilon(x)$. Because $y \in U(x)$, $T(y) \subset V_{\varepsilon(x)}(Tx)$, and hence $p \in V_{\varepsilon(x)}(Tx)$. Consequently, there is a point $b \in T(x)$ such that $d(p, b) < \varepsilon(x)$. Hence $d(b, z) < \eta(x)$, and thus, $z \in F(y) \cap V_{\eta(x)}(Tx) \subset F(U(x)) \cap V_{\eta(x)}(Tx) = \varnothing$. It is a contradiction.

Let $\delta(x) = \sup\{r: 0 < r < \varepsilon$ and $F(x) \cap V_r(Tx) = \varnothing\}$. Obviously, $\delta(x) \leq \varepsilon$ and for each $y \in U(x)$, $\delta(y) \geq \varepsilon(x)$. Now, we assert that $F(x) \cap V_{\delta(x)}(Tx) = \varnothing$. Otherwise, there is a point $y \in F(x)$ and there is a point

$z \in T(x)$ such that $d(y, z) < \delta(x)$. Consequently, there is a number $r > d(y, z)$ such that $0 < r < \varepsilon$ and $F(x) \cap V_r(Tx) = \emptyset$. Hence $y \in F(x) \cap V_r(Tx) = \emptyset$. It is a contradiction.

Because $F: X \to 2^Y$ is quasi-lower semicontinuous, there exist a point $y_x \in F(x)$ and an open neighborhood $N(x)$ of x in X such that $N(x) \subset U(x)$ and

$$F(z) \cap V_{\varepsilon(x)}(y_x) \neq \emptyset, \qquad \forall z \in N(x).$$

Because again X is paracompact, there is a locally finite open refinement $\mathscr{R}$ of $\{N(x): x \in X\}$. For each $O \in \mathscr{R}$, taking a fixed $z(O) \in X$ such that $O \subset N(z(O))$. Let $\eta(O) = y_{z(O)}$. Then $\eta: \mathscr{R} \to Y$ is a mapping. By Theorem 1 in [3, p. 346] there is a continuous mapping $f: X \to Y$ such that $f(x) \in \Gamma_{\{\eta(O):\, O \in \sigma(x, \mathscr{R})\}}$, where $\sigma(x, \mathscr{R}) = \{O \in \mathscr{R}: x \in O\}$.

For each $x \in X$ and for each $O \in \sigma(x, \mathscr{R})$, we have $F(x) \cap V_{\varepsilon(z(O))}(\eta(O)) \neq \emptyset$ and $\delta(x) \geq \varepsilon(z(O))$ because $x \in O \subset N(z(O)) \subset U(z(O))$, and hence $F(x) \cap V_{\delta(x)}(\eta(O)) \neq \emptyset$. Consequently,

$$f(x) \in \Gamma_{\{\eta(O):\, O \in \sigma(x, \mathscr{R})\}} \subset \left\{y \in Y: F(x) \cap V_{\delta(x)}(y) \neq \emptyset\right\},$$

because $F(x)$ is H-convex. Hence $F(x) \cap V_{\delta(x)}(f(x)) \neq \emptyset$, and hence

$$F(x) \cap V_{\varepsilon}(f(x)) \neq \emptyset \quad \text{and} \quad f(x) \notin T(x), \qquad \forall x \in X.$$

This completes the proof.

Remark. Theorem 3.7 improves and generalizes Theorem 2.3 of [17] to an a.l.c. metric space.

4. ALMOST FIXED POINT THEOREM AND FIXED POINT THEOREM

THEOREM 4.1. *Let $(X, \{\Gamma_A\}, \mathscr{U})$ be a Hausdorff l.c.-space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, where $\mathscr{U}$ is the uniform structure on X. Let D be a nonempty H-compact subset of X and let $T: X \to 2^D$ be a quasi-lower semicontinuous multivalued mapping. Then for each $U \in \mathscr{U}$, there exists a point $\bar{x} \in X$ such that H-co $T(\bar{x}) \cap U(\bar{x}) \neq \emptyset$.*

Proof. Because D is nonempty H-compact subset of X, there exists a compact weakly H-convex subset E of X such that $D \subset E$. Because again $T: X \to 2^D$ is a quasi-lower semicontinuous multivalued mapping, so is the mapping $S: E \to 2^E$ defined by

$$S(x) = H\text{-co}(T(x)) \cap E, \qquad \forall x \in E.$$

For each $U \in \mathscr{U}$, by Theorem 3.1 there exists a continuous mapping $f: E \to E$ such that $S(x) \cap U(f(x)) \neq \emptyset$ for all $x \in E$.

Note that $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, $(X, \{\Gamma_A\}, \mathscr{U})$ is a locally convex H-space. By virtue of Lemma 2, there exists a point $\bar{x} \in E$ such that $\bar{x} = f(\bar{x})$. hence H-co $T(\bar{x}) \cap U(\bar{x}) \neq \emptyset$. This completes the proof.

THEOREM 4.2. *Let $(X, \{\Gamma_A\}, \mathscr{U})$ be a Hausdorff l.c.-space with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, where $\mathscr{U}$ is the uniform structure on X. Let D be a nonempty H-compact subset of X and let $T: X \to 2^D$ be a quasi-lower semicontinuous multivalued mapping with nonempty H-convex values. If the graph* $\mathrm{Gr}(T)$ *of T is closed in $X \times X$, then there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.*

Proof. Because D is a nonempty H-compact subset of X, there exists a compact weakly H-convex subset E of X such that $D \subset E$. Let $\mathscr{B}$ be the family of all closed symmetric elements in $\mathscr{U}$. Then $\mathscr{B}$ is a base of $\mathscr{U}$. By the proof of Theorem 4.1 we know that for each $U \in \mathscr{B}$, there is a point $x_U \in E$ such that $x_U \in U(T(x_U))$. Let $Q_U = \{x \in E: x \in U(T(x))\}$. Then Q_U is a nonempty subset of E. Let $\{x_\alpha: \alpha \in I\}$ be a net in Q_U such that $x_\alpha \to x \in E$. Then for each $\alpha \in I$, there exists a point $y_\alpha \in T(x_\alpha)$ such that $(y_\alpha, x_\alpha) \in U$. Because the set E is compact, we may assume $y_\alpha \to y$. Because again U and $\mathrm{Gr}(T)$ are closed in $X \times X$, $(y, x) \in U$, and $y \in T(x)$, and so $x \in U(T(x))$. This shows that the set Q_U is closed in E.

For any finite elements $U_1, U_2, \dots, U_n$ of $\mathscr{B}$, because $\mathscr{B}$ is a base of $\mathscr{U}$, there is an element $U_0 \in \mathscr{B}$ such that $U_0 \subset \bigcap_{i=1}^n U_i$. Consequently, $\bigcap_{i=1}^n U_i \neq \emptyset$. This shows the $\{Q_U: U \in \mathscr{B}\}$ has finite intersection property. Hence $\bigcap_{U \in \mathscr{B}} Q_U \neq \emptyset$ because E is compact. Take any $\bar{x} \in \bigcap_{U \in \mathscr{B}} Q_U$. Then $\bar{x} \in \bigcap_{U \in \mathscr{B}} U(T(\bar{x}))$. Note that $\mathrm{Gr}(T)$ is closed in $X \times E$. By Lemma 1 we know that $\bar{x} \in T(\bar{x}) \subset D$. This completes the proof.

Remark. Theorem 4.1 is a new almost fixed point theorem, which is different from the almost fixed point theorems in [5–7]. Theorem 4.2 improves and extends the well-known Brouwer–Schauder–Tychonoff fixed point theorem (see [11, 15]), and it is different from Theorem 2.1 in [14].

5. A NEW EXISTENCE THEOREM FOR THE SOLUTION OF THE GENERALIZED QUASIVARIATIONAL INEQUALITY

THEOREM 5.1. *Let $(X, \{\Gamma_A\}, \mathscr{U})$ be a Hausdorff locally convex H-space and let $(Y, \{\hat{\Gamma}_A\}, \mathscr{B})$ be an l.c.-space, where $\mathscr{U}$ is the uniform structure on X and $\mathscr{B}$ is the uniform structure on Y. Let D be a nonempty H-compact subset of X, $S: X \to 2^D$ a continuous multivalued mapping with nonempty closed values and let $T: X \to 2^Y$ be a quasi-lower semicontinuous multivalued* *mapping with* H*-convex values. If a continuous function* $\phi\colon X \times Y \times X \to R$ *such that*

(i) *for each* $x \in X$ *and for each* $y \in Y$, *the set*

$$\left\{ z \in S(x) \colon \phi(x, y, z) = \min_{u \in S(x)} \phi(x, y, u) \right\}$$

is acyclic;

(ii) *for each* $x \in X$ *and for each* $y \in Y$, $\phi(x, y, x) \geq 0$.

Then

(1) *for each* $V \in \mathscr{B}$, *there exist a point* $\bar{x} \in S(\bar{x})$ *and a point* $\bar{y} \in V(T(\bar{x}))$ *such that*

$$\phi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}).$$

(2) *if, in addition,* Y *is compact and* T *is upper semicontinuous with closed values, then there exists a point* $\bar{x} \in S(\bar{x})$ *and a point* $\bar{y} \in T(\bar{x})$ *such that*

$$\phi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}).$$

Proof (1). Because D is a nomempty H-compact subset of X, there exists a compact weakly H-convex subset E of X such that $D \subset E$. Consequently, $(E, \{E \cap \Gamma_A\})$ is also a Hausdorff locally convex H-space. For each $V \in \mathscr{B}$, take an open symmetric element $U \in \mathscr{B}$. By Theorem 3.1 there exists a continuous mapping $f\colon E \to Y$ such that $f(x) \in U(T(x))$ for all $x \in E$.

For each $x \in E$, let

$$H(x) = \left\{ z \in S(x) \colon \phi(x, f(x), z) = \min_{u \in S(x)} \phi(x, f(x), u) \right\}.$$

Because $S\colon E \to 2^D$ is a continuous multivalued mapping with nonempty compact values, $H\colon E \to 2^D$ is an upper semicontinuous multivalued mapping by Proposition 23 in Aubin−Ekeland [1, p. 120]. Moreover, by (i) and the continuity of ϕ we know that $H(x)$ is a closed acyclic subset of D. By virtue of Lemma 2, there exists a point $\bar{x} \in D$ such that $\bar{x} \in H(\bar{x})$, i.e., $\bar{x} \in S(\bar{x})$ and

$$\phi(\bar{x}, f(\bar{x}), \bar{x}) = \min_{u \in S(\bar{x})} \phi(\bar{x}, f(\bar{x}), u).$$

Now, taking $\bar{y} = f(\bar{x})$, then $\bar{y} \in V(T(\bar{x}))$ and

$$\min_{u \in S(\bar{x})} \phi(\bar{x}, \bar{y}, u) = \phi(\bar{x}, \bar{y}, \bar{x}) \geq 0,$$

by (ii), i.e.,

$$\phi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}).$$

Proof (2). Let $\mathscr{V} = \{V \in \mathscr{B}: V$ is closed and symmetric$\}$. Then $\mathscr{V}$ is a base of $\mathscr{B}$. For each $V \in \mathscr{V}$, by (1) there exist a point $x_V \in E$ and a point $y_V \in Y$ such that $x_V \in S(x_V)$, $y_V \in V(T(x_V))$ and

$$\phi(x_V, y_V, x) \geq 0, \qquad \forall x \in S(x_V).$$

For each $x \in X$, let $F(x) = V(T(x))$. Then F: $X \to 2^Y$ is a set-valued mapping. If a net $\{(x_\alpha, y_\alpha)\}_{\alpha \in I} \subset \mathrm{Gr}(F)$ ($\mathrm{Gr}(F)$ denotes the graph of F) such that $x_\alpha \to x, y_\alpha \to y$, then for each $\alpha \in I$, there exists $z_\alpha \in T(x_\alpha)$ such that $(y_\alpha, z_\alpha) \in V$. By Proposition 1 of Su−Sehgal [10] there exist $z \in T(x)$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \to z$. consequently, $(y, z) \in V$ because V is closed. Hence $y \in F(x)$. It shows that $\mathrm{Gr}(F)$ is closed. Now, let

$$Q_V = \Big\{(x, y) \in E \times Y: x \in S(x),\ y \in V(T(x)) \text{ and } \min_{u \in S(x)} \phi(x, y, u) \geq 0\Big\}.$$

Then Q_V is a nonempty closed subset of $E \times Y$. Obviously, $\{Q_V: V \in \mathscr{V}\}$ has a finitely intersection property. Hence,

$$\bigcap_{V \in \mathscr{V}} Q_V \neq \varnothing.$$

Consequently, there is a point,

$$(\bar{x}, \bar{y}) \in \bigcap_{V \in \mathscr{V}} Q_V,$$

and hence $\bar{x} \in S(\bar{x})$,

$$\phi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}),$$

and

$$\bar{y} \in \bigcap_{V \in \mathscr{V}} V(T(\bar{x})).$$

By Lemma 1 we know that $\bar{y} \in T(\bar{x})$. This completes the proof.

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